Journal of Number Theory 130 (2010) 534–538

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Sets whose sumset avoids a thin sequence

Vishaal Kapoor

University of British Columbia, Department of Mathematics, Room 121, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada

ARTICLE INFO

Article history:
Received 12 February 2008
Available online 25 November 2009
Communicated by Ronald Graham

Keywords:
Additive bases
Sumsets
Sequences

ABSTRACT

Let $\{a_1, a_2, a_3, \ldots\}$ be an unbounded sequence of positive integers with a_{n+1}/a_n approaching α as $n \to \infty$, and let $\beta > \max(\alpha, 2)$. We show that for all sufficiently large $x \geqslant 0$, if $A \subset [0, x]$ is a set of nonnegative integers containing 0 and satisfying

$$|A| \geqslant \left(1 - \frac{1}{\beta}\right)x,$$

then we can represent some element of the sequence $\{a_n\}$ as a pairwise sum of elements of A. We also prove an analogous result which holds for all $x \geqslant 0$.

In [1], Erdős and Freud conjectured if $A \subset \{1, 2, \ldots, 3n\}$ is a set of at least $n+1$ elements then there is some power of two that could be written as the sum of distinct elements of A. This was proved by Erdős and Freiman [2] for sufficiently large n, and later improved by Nathanson and Sárközy [4].

We could also ask what happens if we restrict the number of summands. Lev [3] showed that if $A \subset \{0, 1, 2, \ldots, n\}$ contains 0 and has at least $n/2 + 1$ elements, then a power of 2 can be written as the sum of two elements of A. And more recently, Pan [5] proved the essentially sharp result (for $m \geqslant 3$): if $0 \in A$, then $|A| \geqslant (1 - 1/m)n + 1$ implies that a power of m can be written as the sum of two elements in A.

In this paper, we consider sequences that grow essentially like the powers of a real number greater than or equal to 2. Given a set A, denote by $2A$ the set of pairwise sums $\{a_1 + a_2\colon a_1, a_2 \in A\}$. (Note that this set is usually denoted by $A + A$.) We then have the following result:

E-mail address: vkapoor@math.ubc.ca.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.09.018

Theorem 1. *Let $\{a_1, a_2, a_3, \ldots\}$ be an unbounded sequence of positive integers. Assume that a_{n+1}/a_n approaches some limit α as $n \to \infty$, and let $\beta > 2$ be some real number greater than α. Then for sufficiently large $x \geqslant 0$, if A is a set of nonnegative integers less than or equal to x containing 0 and satisfying*

$$|A| \geqslant \left(1 - \frac{1}{\beta}\right)x, \tag{1}$$

then $2A$ contains an element of $\{a_n\}$.

We remark that the sequences under consideration in Theorem 1 include a large class of recurrence sequences. In particular, it includes powers of integers $\geqslant 2$ as previously considered.

Theorem 1 follows from the more general result:

Theorem 2. *Let $\{a_1, a_2, a_3, \ldots\}$ be an unbounded sequence of positive integers such that $a_{n+1}/a_n \leqslant \beta$ for some constant $\beta \geqslant 2$. Then for any $x \geqslant 0$, if A is a set of nonnegative integers less than or equal to x containing 0 and satisfying*

$$|A| > \left(1 - \frac{1}{\beta}\right)x + \frac{1}{\beta} \cdot \left\lfloor \frac{a_1 - 1}{2} \right\rfloor + 1, \tag{2}$$

then $2A$ contains an element of $\{a_n\}$.

We remark that if $a_1 = 1$, the strict inequality can be replaced by the nonstrict inequality $\geqslant$ as long as $\beta > 2$ and x is a positive number. We will return to this point following the proof of Theorem 2.

For sequences with $\lim a_{n+1}/a_n = \alpha \geqslant 2$, Theorem 1 is sharp in the sense that we cannot take β to be less than or equal to α. Indeed, let $\{a_n\}$ be a sequence such that $a_{n+1}/(a_n + 2)$ tends to α from above, and define the sets

$$A_n = \{0\} \cup (a_n/2, a_n) \cup (a_n, a_{n+1}/2) \quad (n \in \mathbb{N}).$$

We see that

$$\lim a_{n+1}/a_n = \frac{\lim a_{n+1}/a_n}{\lim(1 + 2/a_n)} = \lim \frac{a_{n+1}}{a_n + 2} = \alpha,$$

as a_n tends to infinity. Moreover, for $x = a_{n+1}/2$,

$$|A_n| \geqslant \frac{a_{n+1} - a_n}{2} - 1 \geqslant \left(1 - \frac{a_n + 2}{a_{n+1}}\right)x > \left(1 - \frac{1}{\alpha}\right)x.$$

However, $2A_n$ is a subset of $\{0\} \cup (a_n/2, a_n) \cup (a_n, a_{n+1})$, which does not contain any element of $\{a_n\}$ as $a_{n+1} \geqslant 2a_n$. Thus any choice of β must be greater than α.

The proof of Theorem 2 proceeds along the lines of the proof of Lev's lemma [3] about powers of 2.

Proof of Theorem 2. First let us suppose that $A \subset [0, x]$ is a set satisfying the hypotheses of the theorem for some $0 \leqslant x < a_1$. Set $x_0 = \lfloor (a_1 - 1)/2 \rfloor$. If $x < x_0$ then $|A| > (1 - 1/\beta)x + x_0/\beta + 1 > x + 1$, and so the theorem is vacuously true. If $x_0 \leqslant x < a_1$, then $|A| > (1 - 1/\beta)x + x_0/\beta + 1 \geqslant x_0 + 1 = \lfloor (a_1 + 1)/2 \rfloor$. Thus A and $a_1 - A$ are both subsets of $[0, a_1]$ of size $> \lfloor (a_1 + 1)/2 \rfloor$. By the pigeonhole principle they have a common element – hence $a_1 \in 2A$. Therefore the theorem holds for $0 \leqslant x < a_1$.

We now proceed by induction on $\lfloor x\rfloor$. Assume $x \geqslant a_1$ is a nonnegative integer, and choose r so that $a_r \leqslant x < a_{r+1}$. Let A be a set satisfying the hypotheses of the theorem; and suppose, for sake of contradiction, that $2A$ is disjoint from $\{a_n\}$. We have two cases:

Case $x \leqslant a_{r+1}/2$: Partition A into the sets $A_1 = A \cap [0, a_r]$ and $A_2 = A \cap (a_r, x]$. The sets A_1 and $a_r - A_1$ together lie in $[0, a_r]$ and must be disjoint, or else $2A_1$ would contain an element of $\{a_n\}$. This pairing argument shows that $|A_1| \leqslant (a_r + 1)/2$.

Combining this estimate with the trivial estimate $|A_2| \leqslant x - a_r$ shows that A can have no more than $x - a_r + (a_r + 1)/2 \leqslant x(1 - a_r/2x) + 1/2$ members. This upper bound is largest when $x = \lfloor a_{r+1}/2\rfloor \leqslant a_{r+1}/2$ and so we see that $|A| \leqslant x(1 - a_r/a_{r+1}) + 1/2$. But this is a contradiction as $|A| > x(1 - a_r/a_{r+1}) + 1/2$. Thus $2A$ contains an element of $\{a_n\}$.

Case $x > a_{r+1}/2$: Define $A_1 = A \cap [0, a_{r+1} - x)$ and $A_2 = A \cap [a_{r+1} - x, x]$. Then $a_{r+1} - A_2$ and A_2 both lie in the interval $[a_{r+1} - x, x]$ and must be disjoint. By this pairing argument, we see that A_2 can have no more than $(2x - a_{r+1} + 1)/2$ elements.

Define $c = 1/\beta \cdot \lfloor(a_1 - 1)/2\rfloor + 1$. As $a_{r+1} - x < x$, it follows by induction that if $|A_1| > (1 - 1/\beta)(a_{r+1} - x - 1) + c$, then $2A$ contains an element of $\{a_n\}$.

But $|A| > (1-1/\beta)x + c \geqslant (1-1/\beta)(a_{r+1} - x - 1) + (1-1/2)(2x - a_{r+1} + 1) + c$, the second inequality following as $\beta \geqslant 2$. Thus either $|A_1| > (1 - 1/\beta)(a_{r+1} - x - 1) + c$ or $|A_2| > (2x - a_{r+1} + 1)/2$, giving a contradiction either way. Hence A contains an element of $\{a_n\}$. □

In Theorem 2, if $a_1 = 1$, $\beta > 2$, and $x > 0$, we can replace the hypothesis $|A| > (1 - 1/\beta)x + \lfloor(a_1 - 1)/2\rfloor + 1$ with $|A| \geqslant (1 - 1/\beta)x + 1$. The proof proceeds analogously to the proof of Theorem 2, except for the special case when $x = a_{r+1} - 1$. As this new theorem does not hold for $x = 0$, if $x = a_{r+1} - 1$, we cannot apply the induction hypothesis to the set $A_1 = \{0\} \subset [0, a_{r+1} - x)$. However, we instead note that A and $a_{r+1} - A$ are disjoint sets in the interval $[0, a_{r+1}]$, so $|A| \leqslant (a_{r+1} + 1)/2$. As $\beta > 2$, we see that $|A| \geqslant (1 - 1/\beta)(a_{r+1} - 1) + 1 > (a_{r+1} + 1)/2$ which gives the required contradiction.

Theorem 1 follows from Theorem 2 as follows: Let β^- be some constant satisfying $\alpha < \beta^- < \beta$, and assume that

$$\frac{a_{n+1}}{a_n} \leqslant \beta^- \quad \text{for all } n \geqslant 1. \tag{3}$$

Then for any $x \geqslant 0$ so large that $(1 - 1/\beta)x \geqslant (1 - 1/\beta^-)x + 1/\beta \cdot \lfloor(a_1 - 1)/2\rfloor + 1$, we see that Theorem 2, using the constant β^-, gives the conclusion of Theorem 1. If (3) does not hold for all $n \geqslant 1$, then as $a_{n+1}/a_n \leqslant \beta^-$ for sufficiently large n, a simple relabeling of the terms of the sequence, omitting finitely many terms at the beginning, would suffice.

Given a strictly increasing sequence $\{a_1 < a_2 < a_3 < \cdots\} \subset \mathbb{Z}^+$, let $M(x)$ be the smallest value such that if $A \subset [0, x]$ has cardinality greater than $M(x)$, then $2A$ contains an element of $\{a_n\}$. The proof of Theorem 2 gives us an upper bound for $M(x)$, namely

$$M_1(x) = \begin{cases} x - (a_r - 1)/2 & \text{if } a_r \leqslant x \leqslant a_{r+1}/2, \\ \begin{array}{c} x/\beta + 1/\beta \cdot (\lfloor(a_1 - 1)/2\rfloor + 1) \\ + a_{r+1}(1/2 - 1/\beta) + 1/2 \end{array} & \text{if } a_{r+1}/2 < x < a_{r+1}, \end{cases} \tag{4}$$

where r is chosen so that $a_r \leqslant x < a_{r+1}$.

It is possible to obtain a different upper bound for $M(x)$ without the need for induction, relying purely on pairing arguments on the sets in $A \cap [0, a_r]$ and $A \cap (a_r, x]$.

Proposition 3. *Let $\{a_1, a_2, a_3, \ldots\}$ be a sequence of positive integers such that*

$$2 \leqslant \frac{a_{n+1}}{a_n} \leqslant \beta,$$

for some constant β. Then for any $x > 0$, if $A \subset [0, x]$ is a set of positive integers containing 0 with cardinality $|A| > M_2(x)$, then $2A$ contains an element of the sequence $\{a_n\}$. Here

$$M_2(x) = \begin{cases} x - (a_r - 1)/2 & \text{if } a_r \leqslant x < a_{r+1}/2, \\ (a_{r+1} - a_r)/2 & \text{if } a_{r+1}/2 \leqslant x < a_{r+1} - a_r, \\ x - (a_{r+1} - a_r)/2 + 1 & \text{if } a_{r+1} - a_r \leqslant x < a_{r+1} - (a_r + 1)/2, \text{ and} \\ (a_{r+1} + 1)/2 & \text{if } a_{r+1} - (a_r + 1)/2 \leqslant x < a_{r+1}, \end{cases} \tag{5}$$

where we choose r so that $a_r \leqslant x < a_{r+1}$.

Proof. Let A be a set satisfying the hypotheses for $x > 0$, and choose r so that $a_r \leqslant x < a_{r+1}$. Assume that $2A$ is disjoint from the sequence $\{a_n\}$; we show that $|A| \leqslant M_2(x)$. Define

$$A_1 = A \cap [0, a_r],$$

$$A_2 = A \cap (a_r, a_{r+1} - a_r),$$

and

$$A_3 = A \cap [a_{r+1} - a_r, a_{r+1}].$$

We can use a pairing argument, as in the proof of Theorem 2, to obtain the bound $|A_1| \leqslant (a_1 + 1)/2$. For $a_r \leqslant x < a_{r+1}/2$ we bound A_2 trivially, giving $|A| \leqslant (a_1 + 1)/2 + (x - a_1) = x - (a_r - 1)/2$ (as $A_3 = \emptyset$).

When $a_{r+1}/2 \leqslant x < a_{r+1} - a_r$, we use a pairing argument to bound the number of elements in A_2. Indeed, A_2 and $a_{r+1} - A_2$ are disjoint sets that both lie in the interval $(a_r, a_{r+1} - a_r)$ and so $|A_2| \leqslant (a_{r+1} - 2a_r - 1)/2$. This gives $|A| \leqslant (a_{r+1} - a_r)/2$.

For $a_{r+1} - a_r \leqslant x < a_{r+1} - (a_r + 1)/2$ we bound $|A_3|$ trivially, giving $|A| \leqslant (a_{r+1} - a_r)/2 + x - (a_{r+1} - a_r) + 1 = x - (a_{r+1} - a_r)/2 + 1$.

And lastly, for $a_{r+1} - (a_r + 1)/2 \leqslant x \leqslant a_{r+1}$ we bound $A \subset [0, a_{r+1}]$ as we did $|A_1|$. In particular, A and $a_{r+1} - A$ are disjoint sets in $[0, a_{r+1}]$ and so $|A| \leqslant (a_{r+1} + 1)/2$.

In all cases we have $|A| \leqslant M_2(x)$. □

In Fig. 1, we have the bounds given by Theorem 2 and Proposition 3, for the special case of $a_n = 3^n$ in the interval $81 \leqslant x \leqslant 243$. The line $y = 2x/3 + 1$ is solid, while $M_1(x)$ is broken and $M_2(x)$ is dotted.

It is important to note that $M_2(x)$ does not lie under the line $y = (1 - 1/\beta)x + 1/\beta \cdot \lfloor (a_1 - 1)/2 \rfloor + 1$ for all sequences $\{a_n\}$. However, if $\{a_n\}$ is a sequence such that

$$2 \leqslant \alpha \leqslant \frac{a_{n+1}}{a_n} \leqslant \beta,$$

where α and β satisfy the relationship

$$\alpha \geqslant \frac{\beta - 1}{\beta - 2}, \tag{6}$$

then the "corners" $(a_{r+1} - (a_r + 1)/2, (a_{r+1} + 1)/2)$ of $M_2(x)$ – and hence all values of $M_2(x)$ – lie below $y = (1 - 1/\beta)x + 1/\beta \cdot \lfloor (a_1 - 1)/2 \rfloor + 1$. (In Fig. 1 there is only one such corner and we have labeled it "C".) Indeed, if (6) holds then

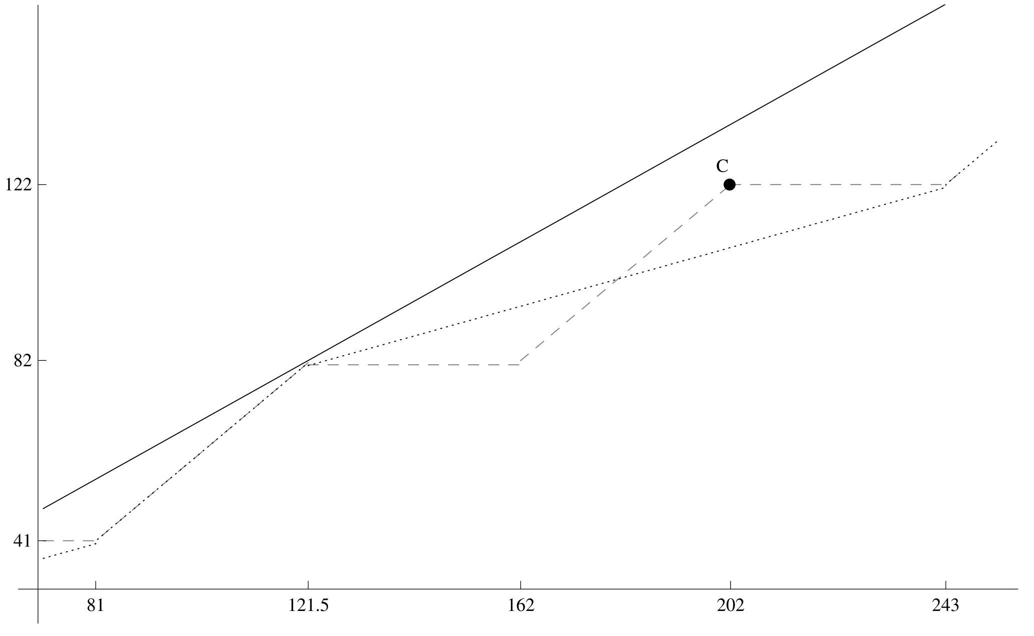

Fig. 1. Upper bounds for $M(x)$.

$$\frac{a_{r+1}}{a_r} \geqslant \alpha \geqslant \frac{\beta-1}{\beta-2} = \frac{(1-1/\beta)}{2(1-1/\beta)-1}.$$

Consequently

$$\left(1-\frac{1}{\beta}\right)\left(a_{r+1}-\frac{a_r}{2}\right) \geqslant \frac{a_{r+1}}{2},$$

which implies that

$$\left(1-\frac{1}{\beta}\right)\left(a_{r+1}-\frac{a_r+1}{2}\right)+\frac{1}{\beta}\cdot\left\lfloor\frac{a_1-1}{2}\right\rfloor+1 > \frac{a_{r+1}+1}{2},$$

as required. Thus for sequences satisfying (6) we have an alternate proof of Theorem 2.

Acknowledgment

I am grateful for the limitless patience and sound advice of my supervisor, Greg Martin.

References

[1] Paul Erdős, Some problems and results on combinatorial number theory, in: Graph Theory and Its Applications: East and West, Jinan, 1986, in: Ann. New York Acad. Sci., vol. 576, New York Acad. Sci., New York, 1989, pp. 132–145.
[2] Paul Erdős, Gregory Freiman, On two additive problems, J. Number Theory 34 (1) (1990) 1–12.
[3] Vsevolod F. Lev, Representing powers of 2 by a sum of four integers, Combinatorica 16 (3) (1996) 413–416.
[4] Melvyn B. Nathanson, András Sárközy, Sumsets containing long arithmetic progressions and powers of 2, Acta Arith. 54 (2) (1989) 147–154.
[5] Hao Pan, Note on integer powers in sumsets, J. Number Theory 117 (1) (2006) 216–221.